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Existence of an equilibrium in a competitive economy with indivisibilities and money

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Abstract

In this paper we introduce a model of an exchange economy with indivisible goods and money. There are finitely many agents each of whom owns one unit of finitely many different types of indivisible goods and a certain amount of money. Each type of indivisible goods is subject to quality differentiation. We demonstrate that under fairly mild conditions on demand the economy has a price equilibrium. The proof is based on a generalization of the well- known lemma of Knaster, Kuratowski and Mazurkiewicz (KKM) in combinatorial topology. The results in the paper generalize those of Gale in the case of just one type of indivisible good present in the economy.

Keywords: Indivisibilities; Equilibria; Combinatorial lemmas

1. Introduction

Since the publication of the seminal article by Gale and Shapley (1962) economic models with one type of indivisible good have been intensively studied by many of researchers, e.g. Shapley and Shubik (1972), Shapley and Scarf (1974), Kelso and Crawford (1982), Kaneko (1982), Quinzii (1984), Gale (1984),

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Kaneko and Yamamoto (1986), and Yamamoto (1987). Gale and Shapley (1962) considered a marriage market with two types of agents: men and women. Each man has preference over women and each woman has preferences over men. Gale and Shapley studied this marked from a game-theoretic point of view and demonstrated that there exists a stable marriage matching. This model was generalized by Kelso and Crawford (1982) as a job matching model by allowing the presence of money. In the models presented by Shapley and Shubik (1972) and Kaneko (1982), there are two types of agents, sellers and buyers, and an indivisible good is traded for money. These models are asymmetric in the sense that sellers and buyers play completely different roles. Shapley and Scarf (1974) presented a similar but symmetric model in which agents exchange their indivisible good without using money. Quinzii (1984) proposed a symmetric model which permits the presence of money and unifies the models above. She proved that the associated cooperative game has a non-empty core and that the associated market has a competitive equilibrium under several conditions on the utility functions of the agents. Gale (1984) made assumptions directly on demands instead of on the utility functions and proposed an elegant proof of the existence of a competitive equilibrium. The proof is based on an intersection lemma, which is a generalization of the (Knaster, Kuratowski and Mazurkiewicz (KKM) lemma. Kaneko and Yamamoto (1986) considered an asymmetric model in which sellers and buyers trade one indivisible good for money. They proved the existence of a competitive equilibrium by using a fixed- point argument. In Yamamoto (1987) the model of Kaneko and Yamamoto is extended by allowing agents initially to own more than one unit of an indivisible good, making no difference between sellers and buyers. he showed, by means of an ingenious argument, that the market has a competitive equilibrium under some conditions on the utility functions of the agents. The crucial assumption of all these models is that there is only one type of indivisible good and each agents demands no more than one unit of the indivisible good. Typically, we may think of the indivisible good as houses, which all have the same function for agents but may differ in quality.

Until now, little progress has been made in dealing with economic equilibrium models with more than one type of indivisible good. Curiel and Tijs (1985) made a step forward by studying two classes of transferable utility games, namely assignment games and permutation games, in which there are two types of indivisible goods. They showed that these games may have an empty core, but they proved that these games have a non-empty core in the case of additivity and separability. In this paper we consider an economy in which there are a finite number of agents and a finite number of different types of indivisible goods. We may think of these different types of indivisible goods as houses, cars, trucks, bikes, and so on. Units of each type of indivisible goods are subject to quality differentiation. For example, houses belong to one type of indivisible goods when they have the same function for agents but may be different in quality. However, houses will be classified into several different types of indivisible goods when they vary in functions. Each agent initially owns at most one unit of each type of indivisible goods and a certain amount of money. Agents have preferences over goods and money with the constraint that no agent desires more than one unit of each type of indivisible goods. We impose rather plausible conditions on demand in order to guarantee the existence of a competitive equilibrium in the economy. The assumptions are similar to those as used by Gale (1984) in the case of just one type of indivisible goods. We only need boundedness and continuity. Our results also applies to a model with some kind of externalities in the sense that whether or not ana gent prefers a good of some type may depend on the prices of the other goods of that type or even on the prices of goods of other types. The proof of the existence of a competitive equilibrium in the economy is based on a generalization of the KKM lemma. The results we obtain in this paper generalize those of Gale and of Curiel and Tijs.

The paper is organized as follows. Section 2 presents the economic model. The proof of the equilibrium existence theorem is given in Section 3.

2. The economic model with indivisibilities and money

We consider an exchange economy with n agents and m different types of indivisible goods. Let I_k be the set of the first k positive integers. We denote the set of agents and the set of different types of indivisible goods by I_n and I_m, respectively. We assume that each agent initially owns one unit of each type of indivisible goods and some amount of money. Notice that this involves no loss of generality since if an agent does not own one unit of some type of indivisible goods we may assume that he has a dummy good of that type which is of no value to any of the agents. We denote that agent i owns one unit of indivisible goods of type j by an ordering pair (j, i). Let $I_m \times I_n$ be the set of all indivisible goods in the economy, i.e. $I_m \times I_n = \{(j, i) \mid j \in I_m, i \in I_n\}$. A price vector is a vector in $\prod_{h=1}^{m} R_+^n$, where R_+^n is the non-negative orthant of the n-dimensional Euclidean space R^n. A vector $p \in \prod_{h=1}^{m} R_+^n$ is represented by $p = (p_1, \ldots, p_m)$, where $p_j = (p_j, \ldots, p_{jn})$ for each $j \in I_m$. Let $\Phi = \{\rho \mid \rho = (\rho_1, \ldots, \rho_n)$ be a permutation of $(1, \ldots, n)\}$. An element $\pi \in \prod_{h=1}^{m} \Phi$ is written as $\pi = (\pi_1, \ldots, \pi_m)$, where $\pi_j = (\pi_j(1), \ldots, \pi_j(n))$ for each $j \in I_m$. Furthermore, for a positive integer k, let I_n^k denote $I_n \times \ldots \times I_n$, where I_n is repeated k times.

For each $i \in I_n$ and each $j \in I_m$, the demand of agent i for the indivisible goods of type j is specified by a covering $C_i^j = \{C_i^{(j,0)}, C_i^{(j,1)}, \ldots, C_i^{(j,n)}\}$ of $\prod_{h=1}^{m} R_+^n$. We remark that each demand set $C_i^{(j,l)}$ is not necessarily a connected set and could be the union of several disjoint subsets. Let us now give some explanation. If $p \in C_i^{(j,k)}$, then this implies that a price p for agent i there is no good of type j owned by any other agent rather than agent k to which he prefers strictly. If $p \in C_i^{(j,0)}$, then this means that at price p for agent i there is no good of type j he prefers above exchanging his good of type j for money.

Definition 2.1. A pair of vectors $(p, \pi) \in \prod_{h=1}^{m} R_+^n \times \prod_{h=1}^{m} \Phi$ is a competitive equilibrium if it holds that $p \in C_i^{(j,\pi_j(i))}$ for all $j \in i_m$ and all $i \in I_n$.

Let B^n denote the boundary of R_+^n. Now we make the following assumptions on the demands:

Assumption 1. The set $C_i^{(j,l)}$ is closed for any $j \in I_m$, $i \in I_n$ and $l \in I_n \cup \{0\}$.

Assumption 2. For each $i \in I_n$ and each $j \in I_m$, $C_i^{(j,1)}, \ldots, C_i^{(j,n)}$ cover $\prod_{h=1}^{m} B^n$.

Assumption 3. There exists $M > 0$ such that if $p_{jl} \geq M$, then $p \notin C_k^{(j,L)}$ for all $k \in I_n$.

Assumption 1 says that the demand of every agent exhibits a certain continuous behavior and that no agent would give up all his money to buy any good. More precisely, it says that if (z^k) is a sequence of price vectors in some demand set $C_i^{(j,l)}$, and if this sequence converges to some price vector z^*, then z^* also lies in the demand set $C_i^{(j,l)}$. Clearly, if each agent has an upper semi-continuous demand correspondence, then Assumption 1 will be satisfied. Assumption 2 says that if the price of a good of some type is zero, then every agent demands some good of that type. In other words, Assumption 2 is a free disposal condition. Finally, Assumption 3 implies that no agent is willing to spend a huge amount of money on any good. It should be noted that the above assumptions are identical to those of Gale (1984, p. 62) in case of just one type of indivisible good present in the economy, i.e. $m = 1$.

We are now ready to state the equilibrium theorem.

Theorem 2.2. *Under Assumptions 1, 2 and 3, the economy has at least one competitive equilibrium* $p,\pi) \in \prod_{h=1}^{m} B^n \times \prod_{h=1}^{m} \Phi$.

3. Proof of the existence of an equilibrium

We first introduce some notation. The vector $e(l)$ is the lth unit vector of R^n for each $l \in I_n$. The vector e denotes a vector in R^n all of whose components equal 1. For each $l \in I_n$, let a^l denote the vector $e/n - e(l)$ in R^n. To prove the equilibrium theorem, we first generalize the KKM lemma. The $(n-1)$-dimensional unit simplex S^n is defined by

$$S^n = \left\{ x \in R_+^n \,\middle|\, \sum_{l=1}^{n} x_l = 1 \right\}$$

Let $\mathscr{S}$ denote the simplotope $\prod_{h=1}^{m} S^n$.

Theorem 3.1. For each $i \in I_n$ and each $j \in I_m$, let the collection of closed sets $\{C_i^{(j,1)}, \ldots, C_i^{(j,n)}\}$ be a covering of the simplotope $\mathcal{S}$ such that if p lies on the boundary of $\mathcal{S}$, then for some $k \in I_n$ it holds that $p \in C_i^{(j,k)}$ and $p_{jk} > 0$. Then there exist $\pi \in \prod_{h=1}^{m} \Phi$ and $p^* \in \mathcal{S}$ such that

$$p^* \in \bigcap_{j=1}^{m} \bigcap_{i=1}^{n} C_i^{(j,\pi_j(i))}.$$

Proof. For each $i \in I_n$ we define

$$C_i^{(i_1,\ldots,i_m)} = \bigcap_{j=1}^{m} C_i^{(j,i_j)}.$$

Clearly, $C_i^{(i_1,\ldots,i_m)}$ is a closed set, and the collection of sets $\{C_i^T \mid T \in I_n^m\}$ is a covering of $\mathcal{S}$. Furthermore, it is not difficult to show that if p lies on the boundary of $\mathcal{S}$, then for some $T = (i_1, \ldots, i_m) \in I_n^m$ it holds that $p \in C_i^T$ and $p_{ji_j} > 0$ for every $J \in I_m$. Now let $\overline{\mathcal{S}}$ denote the set $\mathcal{S} \times S^n$. For each $(i_1, \ldots, i_{m+1}) \in I_n^{m+1}$ we define

$$C^{(i_1,\ldots,i_{m+1})} = C_{i_{m+1}}^{(i_1,\ldots,i_m)} \times S^n.$$

Clearly, $C^{(i_1,\ldots,i_{m+1})}$ is a closed set, and the collection of sets $\{C^T \mid T \in I_n^{m+1}\}$ is a covering of $\overline{\mathcal{S}}$. If p lies on the boundary of $\overline{\mathcal{S}}$, then for some $T = (i_1, \ldots, i_{m+1}) \in I_n^{m+1}$ it holds that $p \in C^T$ and $p_{ji_j} > 0$ for every $j \in I_{m+1}$.

For each $(i_1, \ldots, i_{m+1}) \in I_n^{m+1}$ we define a vector $c^{(i_1,\ldots,i_{m+1})} \in \prod_{h=1}^{m+1} R^n$ by

$$c^{(i_1,\ldots,i_{m+1})} = (a^{i_1}, \ldots, a^{i_{m+1}}).$$

Let the set $V = \prod_{h=1}^{m+1} W^n$ be given where

$$W^n = \left\{ x \in R^n \,\middle|\, \sum_{l=1}^{n} x_l = 1,\ x_l \geq -1/n \text{ for every } l \in I_n \right\}.$$

For $x \in V$ the point $p(x)$ is defined as the projection of x onto $\overline{\mathcal{S}}$, i.e. $p(x) = (p_1(x), \ldots, p_{m+1}(x))$ with the projection $p_h(x)$ of x_h in W^n onto $\mathcal{S}^n$ given by

$$p_{hj}(x) = \begin{cases} 0, & \text{if } x_{hj} < 0, \\ x_{hj} / \sum_{\{l \mid x_{hl} \geq 0\}} x_{hl}, & \text{if } x_{hj} \geq 0, \end{cases}$$

for $h = 1, \ldots, m+1$. Now let the point-to-set mapping F from V to the collection of subsets of $\prod_{h=1}^{m+1} R^n$ be given by

$$F(x) = \mathrm{Conv}\Big(\Big\{ c^{(i_1,\ldots,i_{m+1})} \mid p(x) \in C^{(i_1,\ldots,i_{m+1})} \text{ and } x_{ji_j} \geq 0 \text{ for every } j \in I_{m+1} \Big\}\Big),$$

where Conv(D) denotes the convex hull of a set D. It is easy to see that F is upper semi-continuous. Moreover, $\bigcup_{x \in V} F(x)$ is compact, and for each $x \in V$ the set $F(x)$ is non-empty, convex and compact. Let Y be a compact, convex set in $\prod_{h=1}^{m+1} R^n$ containing $\bigcup_{x \in V} F(x)$. Then we define the point-to-set mapping G from Y to the collection of subsets of V by

$$G(y) = \left\{ x^* \in V \mid x_h^\top y_h \le (x_h^*)^\top y_h \text{ for all } x_h \in W^n \text{ and } h \in I_{m+1} \right\}.$$

Again, G is upper semi-continuous. Moreover, for any $y \in Y$ the set $G(y)$ is non-empty, compact and convex. For $(x, y) \in V \times Y$, let $\phi(x, y)$ be defined as

$$\phi(x, y) = G(y) \times F(x),$$

then ϕ is an upper semi-continuous mapping from the set $V \times Y$ into the collection of non-empty subsets of $V \times Y$ satisfying, for every $(x, y) \in V \times Y$, that the set $\phi(x, y)$ is non-empty, convex and compact. According to Kakutani's fixed point theorem, there exists an $(x^*, y^*) \in V \times Y$ such that

$$x^* \in G(y^*) \quad \text{and} \quad y^* \in \mathrm{F}(x^*).$$

So it holds that

$$x_h^\top y_h^* \le (x_h^*)^\top y_h^*$$

for every $x_h \in W^n$ and $h \in I_{m+1}$. Let β_h be equal to $(x_h^*)^\top y_h^*$. Then by taking x_h equal to e/n, it follows that $\beta_h \ge 0$, since $\sum_{j=1}^n y_{hj}^* = 0$ for any $h \in I_{m+1}$. When we take x_h successively equal to $e(j)$ for every $j \in I_n$, we obtain:

$$y_{hj}^* \le \beta_h \quad \text{for every } h \in I_{m+1} \text{ and } j \in I_n.$$

However, if for some $h \in I_{m+1}$ and $j \in I_n$ it holds that $x_{hj}^* > -1/n$, by taking x_h equal to $x_h^* + \lambda(x_h^* - e(j))$ for arbitrarily small $\lambda > 0$ we obtain that $y_{hj}^* \ge \beta_h$. Hence $y_{hj}^* = \beta_h \ge 0$ when $x_{hj}^* > -1/n$.

Let the collection $\mathscr{T}^*$ of elements of I_n^{m+1} be defined by

$$\mathscr{T}^* = \Big\{ T = (i_1, \ldots, i_{m+1}) \in I_n^{m+1} \mid p(x^*) \in C^T$$
$$\text{and } x_{ji_j}^* \ge 0 \text{ for every } j \in I_{m+1} \Big\}.$$

Suppose that $\mathscr{T}^* = \{T^1, \ldots, T^l\}$, where $T^k = (i_1^k, \ldots, i_{m+1}^k)$. Since $y^* \in F(x^*)$ there exist some non-negative numbers $\mu_1, \ldots, \mu_l$ with sum equal to 1 such that

$$y^* = \sum_{k=1}^{l} \mu_k c^{T^k}.$$

Suppose that $x_{hj}^* = -1/n$ for some $h \in I_{m+1}$ and $j \in I_n$. This it implies that $j \ne i_h^k$ for every $k = 1, \ldots, l$, and hence $y_{hj}^* \ge 0$. Since $\sum_{j=1}^n y_{hj}^* = 0$ for any $h \in I_{m+1}$, we have that $y^* = \mathbf{0}$. So,

$$\sum_{k=1}^{l} \mu_k c^{T^k} = \mathbf{0}. \tag{3.1}$$

Now for each $k \in I_m$ we define

$$S_k = \{(i_k^h, i_{m+1}^h) \mid h = 1, \ldots, l\}.$$

It is clear that S_k is a subset of $I_n \times I_n$. It follows from (3.1) that for every $k \in I_m$,

$$\sum_{(i,j) \in S_k} \mu_{(i,j)}^k (a^i, a^j) = \mathbf{0}$$

and that

$$\sum_{(i,j) \in S_k} \mu_{(i,j)}^k = 1$$

for certain $\mu_{(i,j)}^k \geq 0$ for $(i, j) \in S_k$. Moreover, it holds that for each $i \in I_n$, $\Sigma_j \mu_{(i,j)}^k = 1/n$ and that for each $j \in I_n$, $\Sigma_i \mu_{(i,j)}^k = 1/n$. From this property it follows that the $n \times n$ matrix $U(k)$ with entries $\nu_{(i,j)}^k$ defined by $\nu_{(i,j)}^k = n\mu_{(i,j)}^k$ if $(i, j) \in S_k$ and $\nu_{(i,j)}^k = 0$ if $(i, j) \notin S_k$, is a doubly stoachastic matrix and therefore $U(k)$ is a convex combination of permutation matrices according to the theorem of Birkhoff and von Neumann. So, there exists a permutation $\pi_k = (\pi_k(1), \ldots, \pi_k(n))$ of $(1, \ldots, n)$ such that $\nu_{(\pi_k(j)j)}^k > 0$ and hence $(\pi_k(j), j) \in S_k$ for every $j \in I_n$. Since $p(x^*) \in \bigcap_{h=1}^l C^{T^h}$, it implies that

$$\bigcap_{k=1}^m \bigcap_{j=1}^n C_j^{(i_1^j(k), \ldots, i_{k-1}^j(k), \pi_k(j), i_{k+1}^j(k), \ldots, i_m^j(k))} \neq \emptyset,$$

where $(i_1^j(k), \ldots, i_{k-1}^j(k), \pi_k(j), i_{k+1}^j(k), \ldots, i_m^j(k), j) \in \mathscr{T}^*$ for every $k \in I_m$ and $j \in I_n$. For each $i \in I_n$, since

$$C_i^{(\pi_1(i), \ldots, \pi_m(i))} = \bigcap_{j=1}^m C_i^{(j, \pi_j(i))}$$

we obtain that

$$\bigcap_{j=1}^m \bigcap_{i=1}^n C_i^{(j, \pi_j(i))} \neq \emptyset.$$

This completes the proof. □

We note that the intersection lemma of Gale (1984, p. 63) follows from Theorem 3.1 by setting $m = 1$. The intersection lemma of Gale implies the KKM lemma. Now we are ready to derive Theorem 2.2.

Proof of Theorem 2.2. Without loss of generality we take $M = 1$ for Assumption 3. Let A^n be the intersection of B^n and the unit n-cube U^n. We shall construct a homeomorphism ψ from $\prod_{h=1}^m A^n$ into $\mathscr{S}$ such that for each $j \in I_m$ and $i \in I_n$, the collection $\{\psi(C_i^{(j,k)}) \mid k \in I_n\}$ is a covering of $\mathscr{S}$ satisfying the boundary condition of Theorem 3.1. Then the result immediately follows from Theorem 3.1.

For $p=(p_1,\dots,p_m)\in\prod_{h=1}^{m} A^n$ the point $\psi(p)$ is defined as $\psi(p)=(\psi_1(p_1),\dots,\psi_m(p_m))$, where ψ_h is a homeomorphism from A^n into S^n for each $h\in I_m$. For each $h\in I_m$, the mapping ψ_h is constructed as follows. For each permutation $\rho=(i_1,i_2,\dots,i_n)$ of $(1,\dots,n)$ we define a subset A^ρ of A^n by

$$A^\rho=\left\{p_h\in A^n \mid p_{hi_1}\geq p_{hi_2}\geq \dots \geq p_{hi_n}=0\right\}$$

and define a subset S^ρ of S^n by

$$S^\rho=\left\{x_h\in S^n \mid x_{hi_1}\leq x_{hi_2}\leq \dots \leq x_{hi_n}\right\}.$$

We then define ψ_h from A^ρ to S^ρ by

$$\psi_{hi_1}(p_h)=x_{hi_1}=\frac{1-p_{hi_1}}{n},$$

and

$$\psi_{hi_k}(p_h)=x_{hi_k}=\frac{1-p_{hi_1}}{n}+\frac{p_{hi_1}-p_{hi_2}}{n-1}+\dots+\frac{p_{hi_{k-1}}-p_{hi_k}}{n-k+1},$$

for $k\in I_n\setminus\{1\}$. It is easy to see that $\sum_{k=1}^{n}\psi_{hi_k}(p_h)=1$, and $\psi_{hi_k}(p_h)\geq 0$ for any k. Moreover, $p_h\neq p'_h$ implies $\psi_h(p_h)\neq\psi_h(p'_h)$. Now take $x_h\in S^\rho$. We have

$$x_{hi_1}\leq x_{hi_2}\leq \dots \leq x_{hi_n}.$$

Note that we have

$$p_{hi_1}=1-nx_{hi_1}$$

and

$$p_{hi_k}=p_{hi_{k-1}}-(n-k+1)(x_{hi_k}-x_{hi_{k-1}}),$$

for $k\in I_n\setminus\{1\}$. It follows that $p_h\in A^\rho$. Observe that $\psi_h^{-1}(e(j))=e-e(j)$ and

$$\psi_h^{-1}\left(\left\{x_h\in S^n \mid x_{hj}=0\right\}\right)=\left\{p_h\in A^n \mid p_{hj}=1\right\}.$$

By Assumption 3, $C_i^{(j,k)}$ does not meet $\psi^{-1}(\{x\in\mathscr{S} \mid x_{kk}=0\})$. This implies that $\psi(C_i^{(j,k)})$ does not meet $\{x\in\mathscr{S} \mid x_{jk}=0\}$ for any $i\in I_n$, $j\in I_m$, and $k\in I_n$. Hence it follows that if x is on the boundary of $\mathscr{S}$, then for some $k\in I_n$ it holds that $x\in\psi(C_i^{(j,k)})$ and $x_{jk}>0$. So, the boundary condition of Theorem 3.1 is fulfilled. We obtain the equilibrium theorem. □

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